Dynamics in the neighbourhood of a PtoP connection

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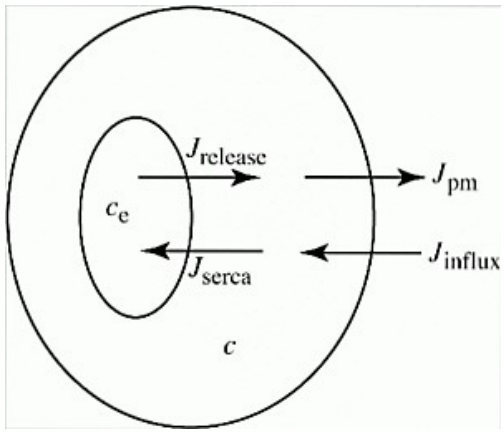
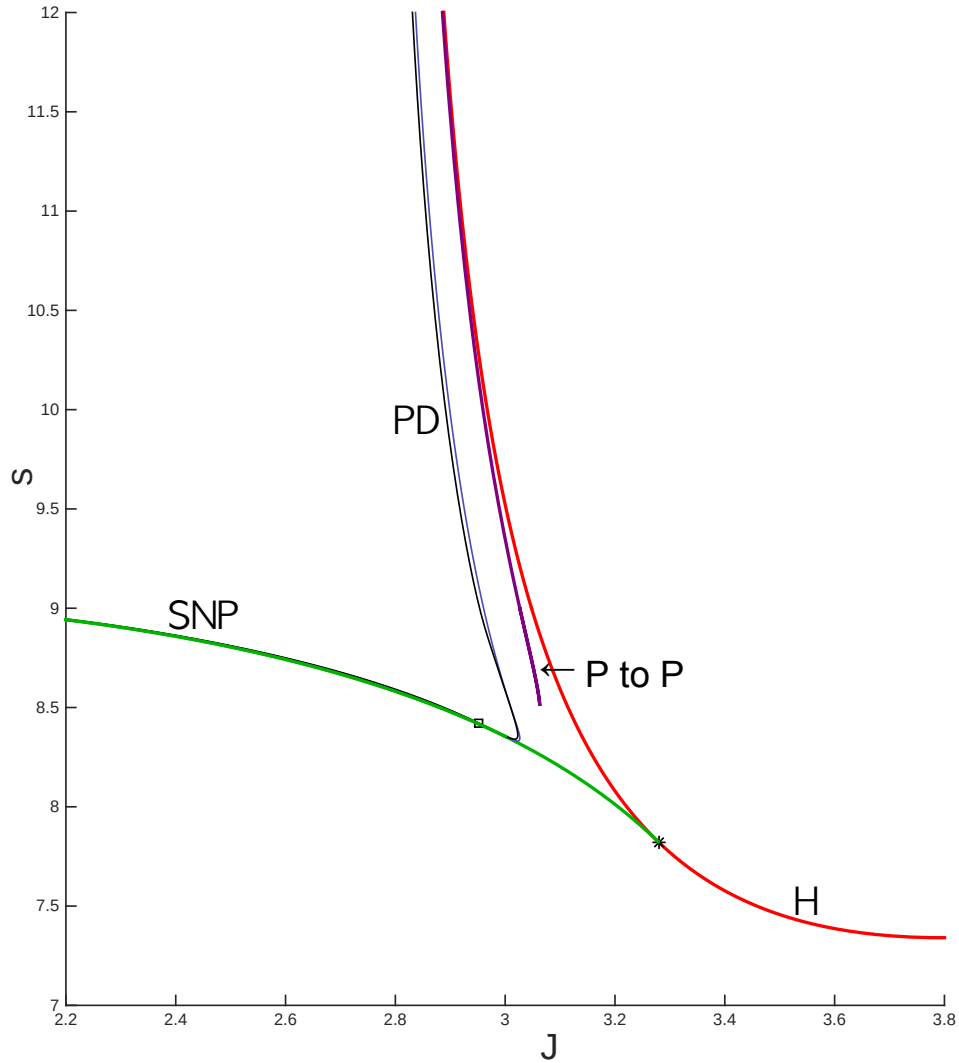
Introduction

This talk will explore the bifurcation structure of a 4-dimensional ODE model for intracellular calcium.

- Extension of an existing 3D ODE model
- Displays a codimension one heteroclinic cycle between periodic orbits – a PtoP cycle.
  - This cannot happen in 3D
  - It gives rise to some interesting surrounding behavior



Bifurcations of a 4D model for intracellular calcium



A four-dimensional calcium model

We consider the following equations:

$$\frac{\mathrm{d}c}{\mathrm{d}t} = v$$

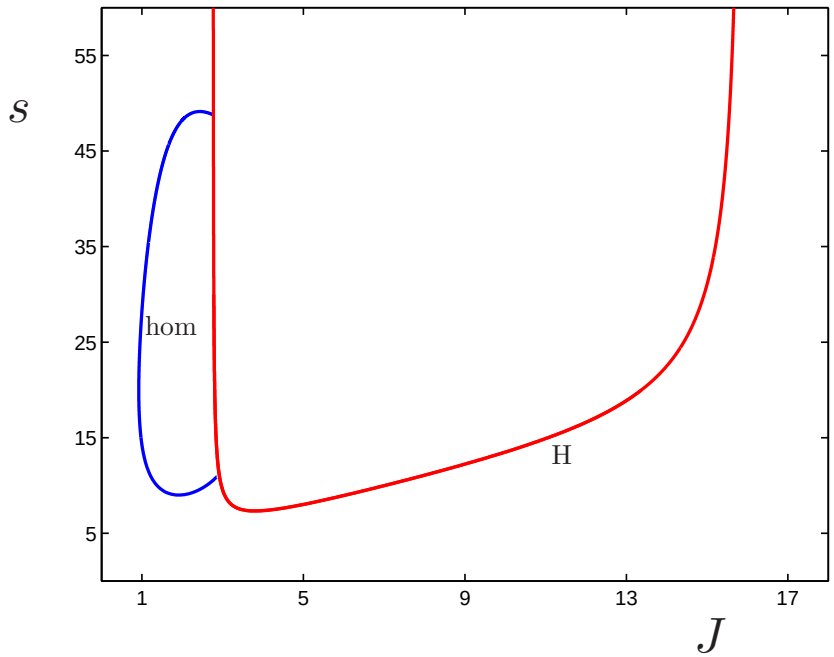
$$D\frac{\mathrm{d}v}{\mathrm{d}t} = sv - \left(\alpha - k_f \frac{c^2}{c^2 - \phi_1^2} n\right)\left(\frac{\gamma(c_t + Dv - sc)}{s} - c\right) + k_s c - \epsilon(J - k_p c)$$

$$\frac{\mathrm{d}c_t}{\mathrm{d}t} = \epsilon\,(J - k_p c)$$

$$s\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\phi_2}{\phi_2 + c} - n\right)$$

(variables in blue, parameters in red)

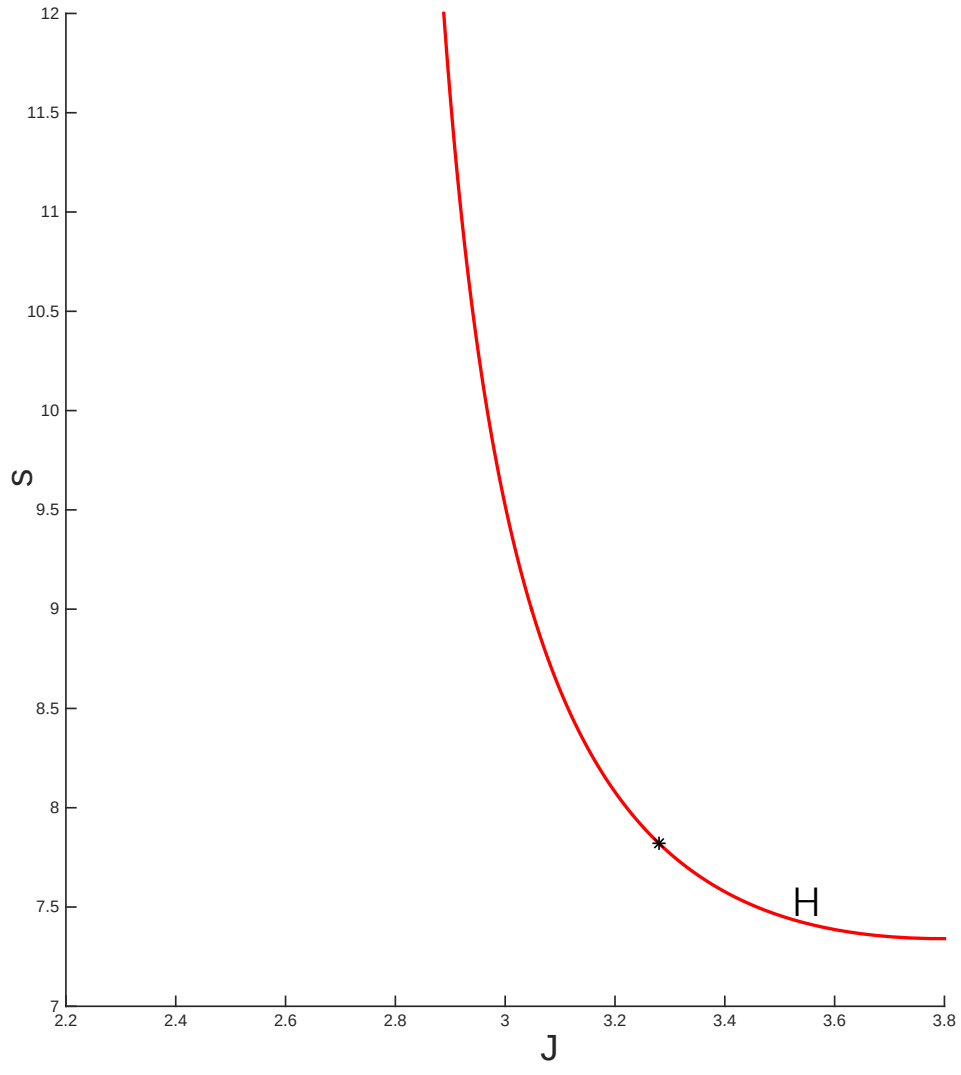
CU Structure



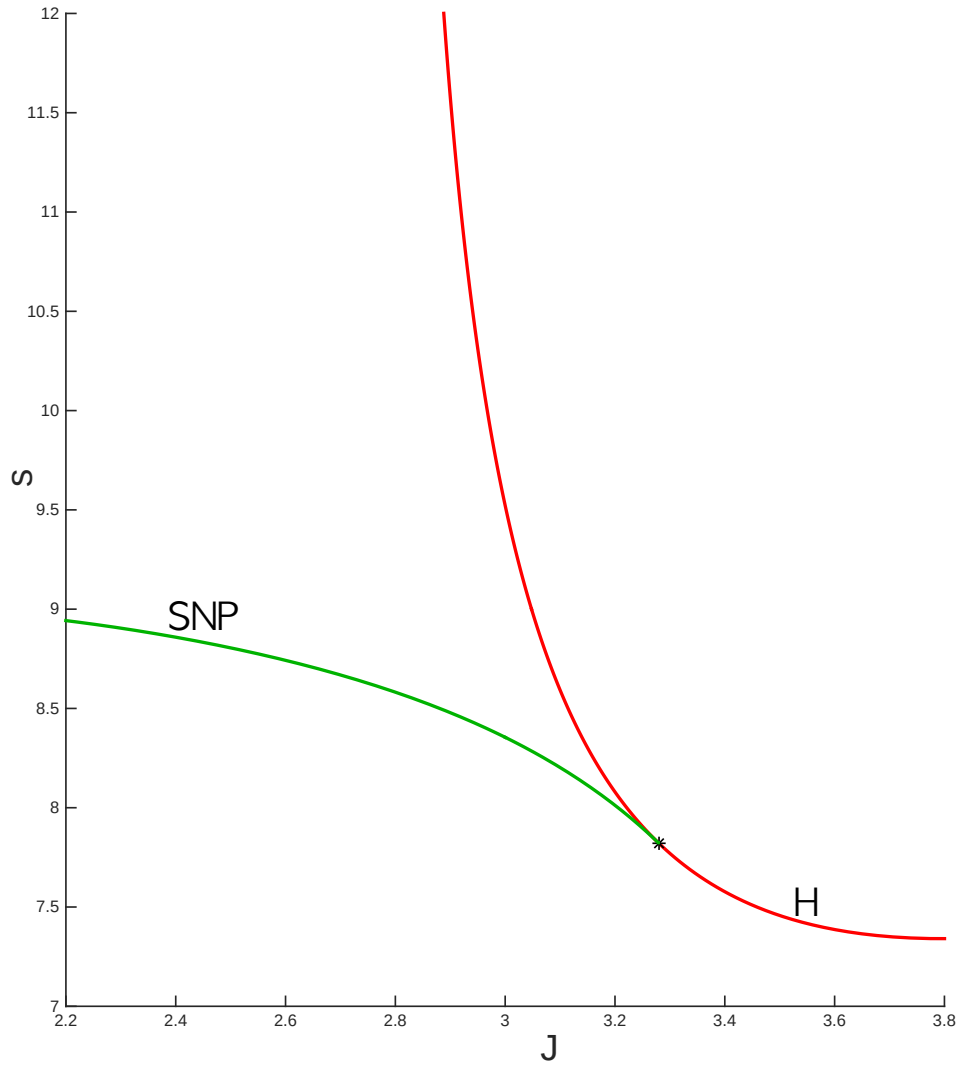
A C shaped homoclinic bifurcation and a U-shaped Hopf curve has been shown to occur in many biological models (e.g. FitzHugh-Nagumo).

When Shil'nikov meets Hopf in Excitable Systems, Champneys et al, 2007

2D parameter plot



2D parameter plot



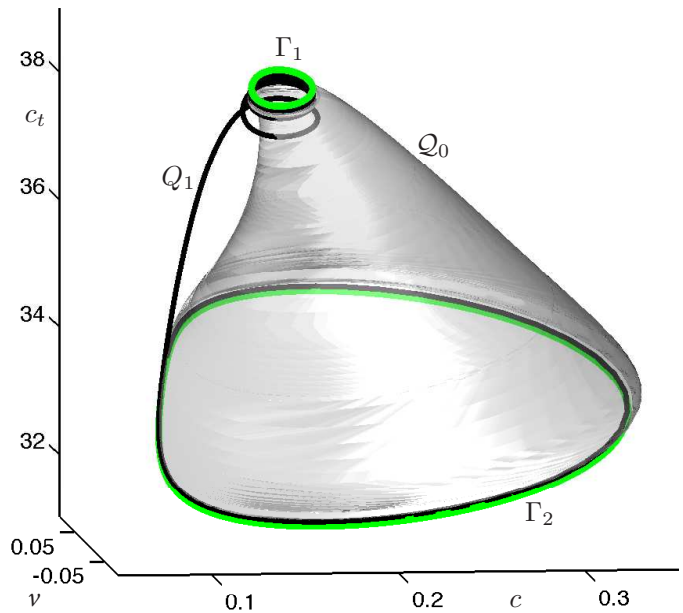
Two periodic orbits

- The green curve shows the location of a *saddle node of periodic orbits* – a place where two periodic orbits meet and disappear.
- Between the red curve and the green curve there exist two periodic orbits.
- Both periodic orbits are of saddle type.
  - One has a 3D stable manifold and a 2D unstable manifold.
  - The other has a 2D stable manifold and a 3D unstable manifold.
- Do these intersect? Do we have paths (i.e. PtoP connections) from one periodic orbit to the other?

PtoP connections

In 4 dimensions, we expect that two 3D manifolds will intersect in a 2D manifold.

- Sort of like a tube between the two periodic orbits.

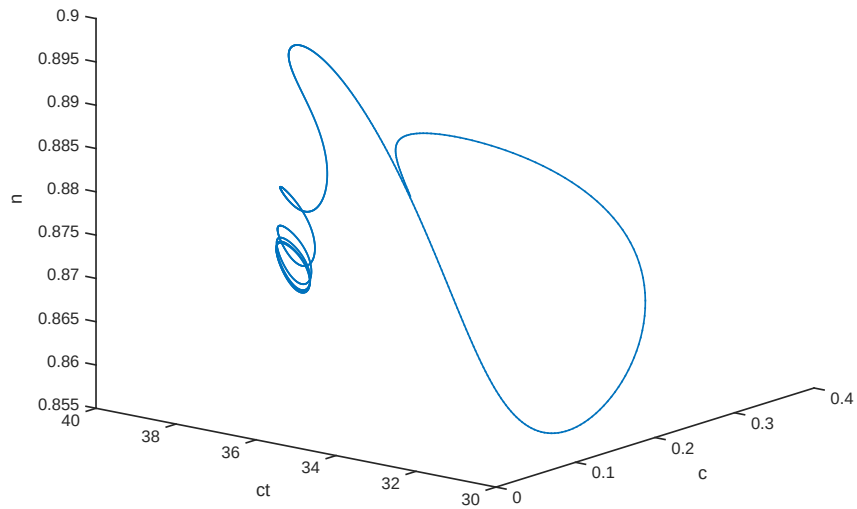


This structure is stable under perturbations.

Codimension-one PtoP connection

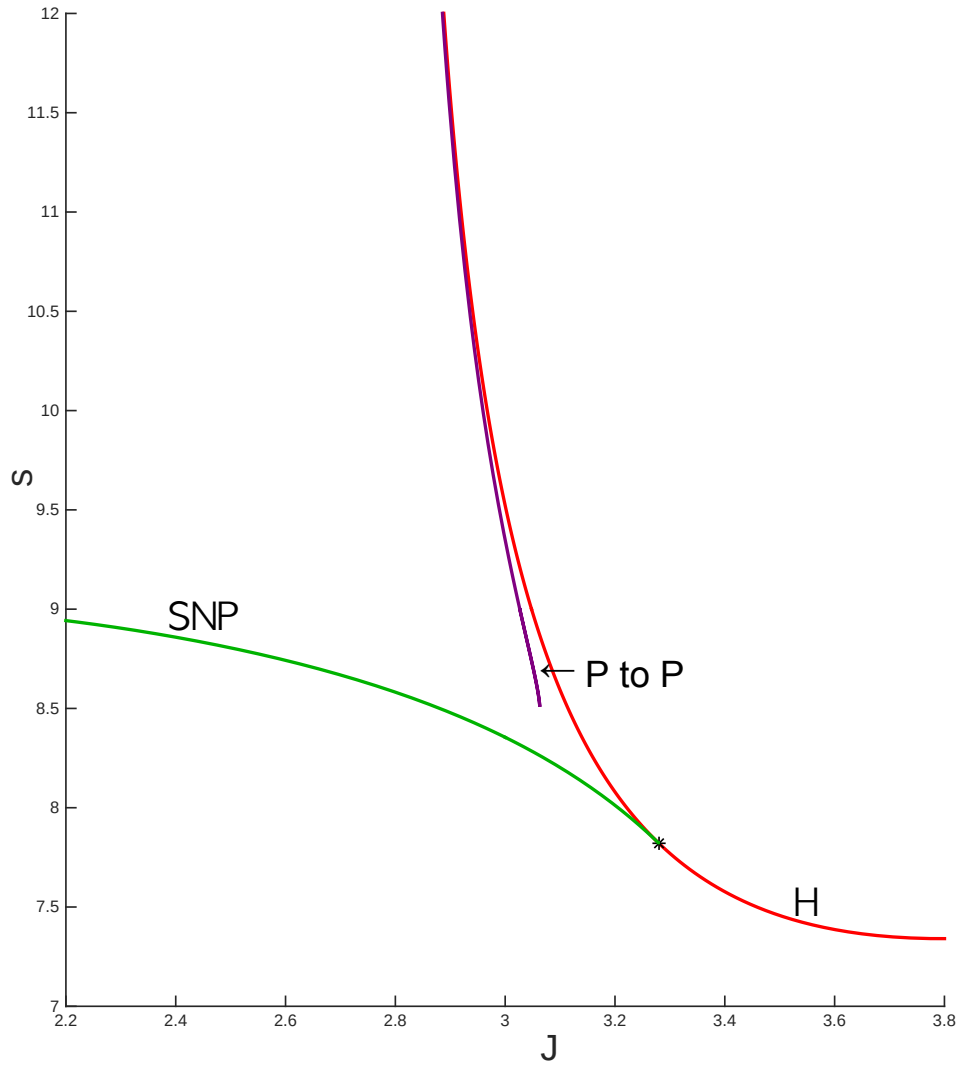
What about the 2D stable/unstable manifolds?

- These intersect in a codimension one bifurcation.
- We use Lin's method to construct a well-posed BVP for this, which we can solve numerically.

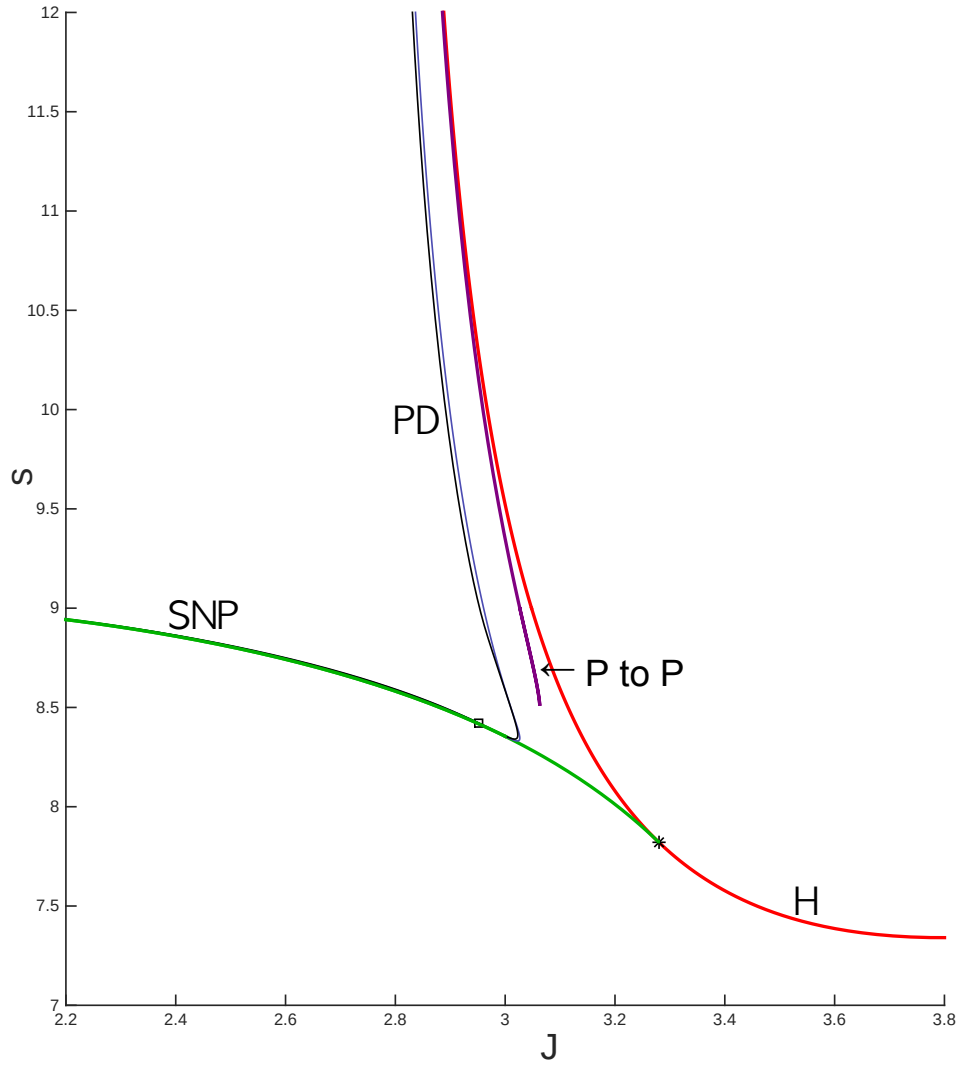


Note that in 3D, a PtoP cycle would have to be stable under perturbations.

2D parameter plot



2D parameter plot



Secondary PtoP cycles?

- We have an *unstable heteroclinic cycle* between the two periodic orbits.
- Nearby, we can find other objects
  - A homoclinic connection from one periodic orbit to itself
  - A periodic orbit close to this homoclinic connection
- Could this new periodic orbit have a heteroclinic PtoP cycle with one of the original two? And so on?

Conclusion

- We have shown that expanding a 3D intracellular calcium model to 4D results in new and interesting dynamics that could not occur in 3D.
- This includes a codimension-one PtoP cycle, which is of theoretical interest.
- In the future, we would like to:
  - Explore the dynamics resulting from and surrounding this PtoP cycle
  - See whether such PtoP cycles occur in other biological models when we use 4 or more dimensions
  - Generalise about the type of chaotic behaviour that occurs in the neighbourhood of codimension one PtoP cycles